Nonautonomous Hamiltonian Systems and Morales–Ramis Theory I. The Case $\ddot{x} = f(x, t)^*$

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Abstract. In this paper we present an approach toward the comprehensive analysis of the nonintegrability of differential equations in the form $\ddot{x} = f(x, t)$ which is analogous to Hamiltonian systems with $1+1/2$ degrees of freedom. In particular, we analyze the nonintegrability of some important families of differential equations such as Painlevé II, Sitnikov, and the Hill–Schrödinger equation. We emphasize Painlevé II, showing its nonintegrability through three different Hamiltonian systems, and also Sitnikov, in which two different versions including numerical results are shown. The main tool for studying the nonintegrability of these kinds of Hamiltonian systems is Morales–Ramis theory. This paper is a very slight improvement to the talk with a similar title delivered by the author at the SIAM Conference on Applications of Dynamical Systems in 2007.

Key words. Hill–Schrödinger equation, Morales–Ramis theory, nonautonomous Hamiltonian systems, nonintegrability of Hamiltonian systems, Painlevé II equation, Sitnikov problem, virtually abelian groups

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1. Introduction. In this section we present the necessary theoretical background for understanding the rest of the paper.

1.1. Differential Galois theory. Our theoretical framework consists of a well-established combination of dynamical systems theory, algebraic geometry, and differential algebra. See [22] or [36] for further information and details. Given a linear differential system with coefficients in $\mathbb{C}(t)$,

$$\dot{z} = A(t) z,$$

(1.1)

a differential field $L \supset \mathbb{C}(t)$ exists, unique up to $\mathbb{C}(t)$-isomorphism, which contains all entries of a fundamental matrix $\Psi = [\psi_1, \ldots, \psi_n]$ of (1.1). Moreover, the group of differential automorphisms of this field extension, called the differential Galois group of (1.1), is an algebraic group $G$ acting over the $\mathbb{C}$-vector space $\langle \psi_1, \ldots, \psi_n \rangle$ of solutions of (1.1) and containing the monodromy group of (1.1).

It is worth recalling that the integrability of a linear system (1.1) is equivalent to the solvability of the identity component $G^0$ of the differential Galois group $G$ of (1.1)—in other words, equivalent to the virtual solvability of $G$.

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It is well established (e.g., [23]) that any linear differential equation system with coefficients in a differential field \( K \),

\[
\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},
\]

by means of an elimination process, is equivalent to the second-order equation

\[
\ddot{\xi} - \left( a(t) + \frac{b(t)}{b(t)} \right) \dot{\xi} - \left( \dot{a}(t) + b(t)c(t) - a(t)d(t) - \frac{a(t)b(t)}{b(t)} \right) \xi = 0,
\]

where \( \xi := \xi_1 \). Furthermore, any equation of the form \( \ddot{z} - 2p\dot{z} - qz = 0 \) can be transformed, through the change of variables \( z = ye^{\int p} \), into \( \ddot{y} = -ry \), \( r \), satisfying the Riccati equation \( \dot{p} = r + q + \rho^2 \). This change is useful since it restricts the study of the Galois group of \( \ddot{y} = -ry \) to that of the algebraic subgroups of \( SL_2(\mathbb{C}) \).

A natural question which now arises is to determine what happens if the coefficients of the differential equation are not all rational. A new method was developed in [3] to transform a linear differential equation of the form \( \ddot{x} = r(t)x \) into an algebraic form—that is, into a differential equation with rational coefficients. This method is called the \textit{algebrization method} and is based on the concept of a \textit{Hamiltonian change of variables} [3]. Such a change is derived from the solution of a one-degree-of-freedom classical Hamiltonian.

\textbf{Definition 1.1 (Hamiltonian change of variables).} A change of variables \( \tau = \tau(t) \) is called \textit{Hamiltonian} if \( (\tau(t), \dot{\tau}(t)) \) is a solution curve of the autonomous Hamiltonian system \( X_H \) with Hamiltonian function

\[
H = H(\tau, p) = \frac{p^2}{2} + \hat{V}(\tau) \text{ for some } \hat{V} \in \mathbb{C}(\tau).
\]

\textbf{Theorem 1.1 (Acosta–Blázquez algebrization method [3]).} Equation \( \ddot{x} = r(t)x \) is algebrizable by means of a Hamiltonian change of variables \( \tau = \tau(t) \) if and only if there exist \( f, \alpha \) such that \( \frac{d}{d\tau} ( \ln \alpha ) \), \( \frac{f}{\alpha} \in \mathbb{C}(\tau) \), where

\[
f(\tau(t)) = r(t), \quad \alpha(\tau) = 2(H - \hat{V}(\tau)) = (\dot{\tau})^2.
\]

Furthermore, the algebraic form of \( \ddot{x} = r(t)x \) is

\[
\frac{d^2 x}{d\tau^2} + \left( \frac{1}{2} \frac{d}{d\tau} \ln \alpha \right) \frac{dx}{d\tau} - \left( \frac{f}{\alpha} \right) x = 0.
\]

The next intended step, once a differential equation has been algebrized, is studying its Galois group and, as a causal consequence, its integrability. Concerning the latter, and by virtue of the invariance of the identity component of the Galois group by finite branched coverings of the independent variable (Morales-Ruiz and Ramis, [25, Theorem 5]), it was proven in [3, Proposition 1] that the identity component of the Galois group is preserved in the algebrization mechanism.
The final step is analyzing the behavior of \( t = \infty \) (or \( \tau = \infty \)) by studying the behavior of \( \eta = 0 \) through the change of variables \( \eta = 1/t \) (or \( \eta = 1/\tau \)) in the transformed differential equation; i.e., \( t = \infty \) (or \( \tau = \infty \)) is an ordinary point (resp., a regular singular point, an irregular singular point) of the original differential equation if and only if \( \eta = 0 \) is one such point for the transformed differential equation.

### 1.2. Morales–Ramis theory

Everything is considered in the complex analytical setting from now on. The heuristics of the titular theory rest on the following general principle: if we assume system

\[
\dot{z} = X(z)
\]

“integrable” in some reasonable sense, then the corresponding variational equations along any integral curve \( \Gamma = \{ \hat{z}(t) : t \in I \} \) of (1.5), defined in the usual manner

\[
(\text{VE}_\Gamma) \quad \dot{\xi} = X'(\hat{z}(t)) \xi,
\]

must be also integrable—in the Galoisian sense of the last paragraph in section 1.1. We assume \( \Gamma \), a Riemann surface, may be locally parametrized in a disc \( I \) of the complex plane; we may now complete \( \Gamma \) to a new Riemann surface \( \overline{\Gamma} \), as detailed in [25, sect. 2.1] (see also [22, sect. 2.3]), by adding equilibrium points, singularities of the vector field, and possible points at infinity.

The aforementioned “reasonable” sense in which to define integrability if system (1.5) is Hamiltonian is obviously the one given by the Liouville–Arnold theorem, and thus the above general principle does have an implementation.

**Theorem 1.2 (Morales-Ruiz and Ramis [22, 25]).** Let \( H \) be an \( n \)-degree-of-freedom Hamiltonian having \( n \) independent rational or meromorphic first integrals in pairwise involution, defined on a neighborhood of an integral curve \( \Gamma \). Then, the identity component \( \text{Gal}(\text{VE}_\Gamma)^0 \) is an abelian group (i.e., \( \text{Gal}(\text{VE}_\Gamma) \) is virtually abelian).

The disjunctive between meromorphic and rational Hamiltonian integrability in Theorem 1.2 is related to the status of \( t = \infty \) as a singularity for the normal variational equations. More specifically, and besides the nonabelian character of the identity component of the Galois group, in order to obtain Galoisian obstructions to the meromorphic integrability of \( H \) the point at infinity must be a regular singular point of \( (\text{VE}_\Gamma) \). On the other hand, for there to be an obstruction to complete sets of rational first integrals, \( t = \infty \) must be an irregular singular point.

See [25, Corollary 8] or [22, Theorem 4.1] for a precise statement and a proof.

### 1.3. Nonautonomous Hamiltonian systems

Nonautonomous Hamiltonian systems on symplectic manifolds have long been the subject of study and appear in a most natural way in classical mechanics and control theory, e.g., [1], [5], [19], [21], [20], [28], [33], [34].

We consider nonautonomous Hamiltonian systems of the form

\[
H = H(q_1, p_1, t) = \frac{p_1^2}{2} + V(q_1, p_1, t).
\]

\( H \) is a nonautonomous Hamiltonian system with \( 1 + 1/2 \) degrees of freedom. It is well known (e.g., [28]) that (1.6) can be included as a subsystem of the Hamiltonian system with two
degrees of freedom given by

\[
\hat{H} = \hat{H}(q_1, q_2, p_1, p_2) = \frac{p_1^2}{2} + V(q_1, q_2) + p_2,
\]

where \(q_2\) and \(p_2\) are conjugate variables, i.e., \(p_2 = -H + k\), where \(k\) is constant, and \(q_2 = t\). Furthermore, \(p_2\) is easily seen to be the offset or counterbalancing energy of the system [28], [30].

Also worth mentioning are some recent results on canonical transformations in the extended phase space [30], [31], [32], [35] as well as on definitions and consequences of “integrability” under such circumstances or generalizations thereof, even for non-Hamiltonian systems [11], [9], [14], [18] which we will not delve into further at this point.

2. Main results. Consider the differential equation

\[
\ddot{x} = f(x, t),
\]

with particular solution \(x = x(t)\). We will henceforth order our choice of positions as \(q_1 = x\) and \(q_2 = t\), thus yielding a Hamiltonian system given by

\[
H = \frac{p_1^2}{2} - F(q_1, q_2), \quad F_{q_1}(q_1, q_2) = \frac{\partial F(q_1, q_2)}{\partial q_1} = f(q_1, q_2).
\]

Equation (2.1) is obviously equivalent to Hamilton’s equations for \(H\),

\[
\dot{q}_1 = p_1 = H_{p_1}, \quad \dot{p}_1 = -H_{q_1} = f(q_1, q_2);
\]

this nonautonomous Hamiltonian system is included as a subsystem of \(X_{\hat{H}}\) linked to \(\hat{H} := H + p_2\), such as in (1.7). Assuming \(x(t) = q_1(t)\) to be a solution of (2.1) and \(q_2(t) = t\), we obtain an integral curve \(\Gamma = \{z(t)\}\) of \(\hat{H}\), where

\[
z(t) := (q_1(t), q_2(t), p_1(t), p_2(t)) = (q_1(t), t, \dot{q}_1(t), -H(t)).
\]

We may now introduce our first main result.

**Theorem 2.1.** Let \(\Gamma\) be an integral curve of \(X_{\hat{H}}\) such as the one introduced above. If \(X_{\hat{H}}\) is integrable by means of rational or meromorphic first integrals, then the Galois group of

\[
(2.2) \quad \tilde{\xi} = (f_{q_1}(q_1, q_2)|\Gamma) \xi
\]

is virtually abelian.

**Proof.** The Hamiltonian field \(X_{\hat{H}}\) is given by \(X_{\hat{H}} = (p_1, 1, f(q_1, q_2), F_{q_2}(q_1, q_2))^T\). The variational equation \(\text{VE}_\Gamma\) along \(\Gamma\) is

\[
(2.3) \quad \frac{d}{dt} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ f_{q_1}(q_1, q_2) & f_{q_2}(q_1, q_2) & 0 & 0 \\ F_{q_1, q_2}(q_1, q_2) & F_{q_2, q_2}(q_1, q_2) & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} |_{\Gamma}.
\]
More precisely, (2.3) is

\[
\begin{cases}
\dot{\xi}_1 &= \xi_3, \\
\dot{\xi}_2 &= 0, \\
\dot{\xi}_3 &= (f_{q_1}(q_1, q_2)|_{\Gamma}) \xi_1 + (f_{q_2}(q_1, q_2)|_{\Gamma}) \xi_2, \\
\dot{\xi}_4 &= (F_{q_1 q_2}(q_1, q_2)|_{\Gamma}) \xi_1 + (F_{q_2 q_2}(q_1, q_2)|_{\Gamma}) \xi_2.
\end{cases}
\]

(2.4)

Hence \(\xi_2 = k\), where \(k\) is constant. Assuming \(k = 0\), the normal variational equations ((NVE\(\Gamma\)); see [8, sect. 1], [25, sect. 4.3], [22, sect. 4.1.3]) for \(\hat{H}\) are given by

\[
\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ f_{q_1}(q_1, q_2)|_{\Gamma} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix},
\]

and solving (2.5) we can obtain \(\xi_4\). System (2.5) is equivalent to (2.2), where \(\xi = \xi_1\). By virtue of [22, Proposition 4.2], the virtual commutativity of Gal (VE\(\Gamma\)) implies that of Gal (NVE\(\Gamma\)); this, coupled with Theorem 1.2, implies that if \(X_{\overline{\Gamma}}\) is rationally or meromorphically integrable, then the Galois group of (2.2) is virtually abelian.

Corollary 2.2. Suppose the Galois group of the differential equation \(\dot{\xi} = k(t)\xi\) is not virtually abelian. Define \(H := \frac{p_1^2}{2} - k(q_2)\frac{q_2^2}{2}\) and \(\hat{H} := H + p_2\). Then \(X_{\overline{\Gamma}}\) is not integrable by means of meromorphic or rational first integrals.

Remark 1. If the Galois group of the equation \(\dot{\xi} = k(t)\xi\) is the Borel group \(G = \mathbb{C}^* \ltimes \mathbb{C}\) (hence connected, solvable, and nonabelian), then \(X_{\overline{\Gamma}}\) is neither meromorphically nor rationally integrable, although it is still possible to solve the equation as well as, ostensibly, the nonautonomous Hamiltonian system \(X_{\overline{\Gamma}}\).

In anticipation of the following corollary, consider, for any \(g(x), a(t),\) and \(\alpha = \alpha_0\) given, the equation

\[
\dot{x} = g_x(x)(g(x) + a(t)) + \alpha, \quad \alpha \in \mathbb{C},
\]

having a certain known particular solution \(q_1 = q_1(t)\). Let

\[H = \frac{p_1^2}{2} - \frac{(g(q_1) + a(q_2))^2}{2} - \alpha q_1, \quad q_2 = t,\]

be a Hamiltonian linked to (2.6) and \(\hat{H} := H + p_2\) its autonomous completion.

Corollary 2.3. If \(X_{\overline{\Gamma}}\) is integrable through rational or meromorphic first integrals, then, along the integral curve \(\Gamma = \{z(t) = (q_1(t), t, q_1(t), -H(t))\}\), the Galois group of the equation

\[
\dot{\xi} = (g_{q_1}^2(q_1) + g_{q_1 q_1}(q_1)g(q_1) + a(q_2))|_{\Gamma} \xi
\]

(2.7)

is virtually abelian.

Now, keeping the above hypotheses for \(g(x), a(t), \alpha = \alpha_0, (2.6),\) and \(x = q_1(t)\), let us define the Hamiltonian system

\[
\hat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - (g(q_1) + a(q_2))p_1 - (\alpha + a_{q_2}(q_2))q_1, \quad \alpha \in \mathbb{C}.
\]

(2.8)
It takes only the following simple calculation to prove that Hamiltonian \( H \) is indeed linked to (2.6) in the manner expected, i.e., as introduced in the beginning of section 2. We have

\[
\begin{align*}
\dot{x} &= g_x(x)(g(x) + a(t)) + \alpha \\
&= g_x(x)\dot{x} + g_x(x)g(x) + g_x(x)a(t) + \alpha + \dot{a}(t) - g_x(x)\dot{x} - \dot{a}(t) \\
&= g_x(x)y + \alpha + \dot{a}(t) - \frac{d}{dt}(g(x) + a(t)),
\end{align*}
\]

where \( y = \dot{x} + g(x) + a(t) \), and requiring \( x \) and \( y \) to be conjugate variables implies the following system, equivalent to (2.6):

\[
\begin{align*}
\dot{x} &= y - (g(x) + a(t)) = H_y, \\
\dot{y} &= g_x(x)y + \alpha + \dot{a}(t) = -H_x,
\end{align*}
\]

A straightforward parallel integration,

\[
H = \frac{y^2}{2} - (g(x) + a(t))y + h_1(x, t) = h_2(y, t) - g(x)y - (\alpha + \dot{a}(t))x,
\]

along with the definition of

\[
h_1(x, t) = -(\alpha + \dot{a}(t))x, \quad h_2(y, t) = \frac{y^2}{2} - a(t)y,
\]

as well as \((q_1, q_2, p_1) = (x, t, y)\), yields the autonomous Hamiltonian system introduced in (2.8).

**Theorem 2.4.** If \( X_H \) is integrable by means of rational or meromorphic first integrals, then, along \( \Gamma = \{ z(t) = (q_1(t), t, p_1(t), -H(t)) \} \), the Galois group of the equation

\[
(2.9) \quad \dot{\xi} = (g_{q_1}^2(q_1) + p_1g_{q_1q_1}(q_1) - g_{q_2q_1}(q_1))\big|_\Gamma \xi
\]

is virtually abelian.

**Proof.** We may proceed as in the proof of Theorem 2.1. The Hamiltonian field \( X_H \) is given by

\[
X_H = \begin{pmatrix}
p_1 - (g(q_1) + a(q_2)) \\
\frac{1}{g_{q_1}(q_1)p_1 + (\alpha + a_{q_2}(q_2))} \\
\frac{a_{q_2}(q_2)p_1 + a_{q_2q_2}(q_2)q_1}{a_{q_2}(q_2)p_1 + a_{q_2q_2}(q_2)q_1}
\end{pmatrix}.
\]

The variational equation \( \text{VE}_\Gamma \) along \( \Gamma = \{(q_1(t), t, p_1(t), -H(t))\} \) is

\[
(2.10) \quad \dot{\xi} = \begin{pmatrix}
\frac{\partial g_{q_1}}{\partial q_1} & -\frac{\partial a(q_2)}{\partial q_2} & 1 & 0 \\
0 & \frac{\partial^2 a(q_2)}{\partial q_2^2} & 0 & 0 \\
\frac{p_1}{\partial^2 a(q_2)} & \frac{\partial a_{q_2}^2}{\partial q_2^2} + q_1 \frac{\partial^3 a(q_2)}{\partial q_2^3} & \frac{\partial g_{q_1}}{\partial q_1} & 0
\end{pmatrix} \big|_\Gamma \xi,
\]
where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)^T$. $\xi_2 \equiv k \in \mathbb{C}$; in particular, $k = 0$ renders system (2.10) equal to the following:

\[
\begin{align*}
\dot{\xi}_1 &= -g_{q_1}(q_1)|_1 \xi_1 + \xi_3, \\
\dot{\xi}_2 &= 0, \\
\dot{\xi}_3 &= p_1 g_{q_1 q_1}(q_1)|_1 \xi_1 + -g_{q_1}(q_1)|_1 \xi_3, \\
\dot{\xi}_4 &= a_{q_2 q_2}(q_2)|_1 \xi_1 + a_{q_2}(q_2)|_1 \xi_3,
\end{align*}
\]

(2.11)

NVE$_\Gamma$ corresponding to

\[
\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} -g_{q_1}(q_1) & 1 \\ p_1 g_{q_1 q_1}(q_1) & g_{q_1}(q_1) \end{pmatrix} \Gamma \begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix}.
\]

(2.12)

Solving (2.12), we obtain $\xi_4$. Using (1.2) and (1.3) system (2.12) is equivalent to (2.9), where $\xi = \xi_1$. Again, by virtue of [22, Proposition 4.2] as in Theorem 2.1, the integrability of Hamiltonian $\hat{H}$ by means of meromorphic or rational first integrals implies the virtual commutativity of the Galois group of (2.9).

**Remark 2.** We observe that if $\ddot{x} = f(x, t)$, and if the same particular solution $x = x(t)$ occurs in the statements of Theorem 2.1, Corollary 2.3, and Theorem 2.4, then we obtain the same NVE$_\Gamma$ ((2.2), (2.7), and (2.9) are equivalent), despite the fact that their respective linked Hamiltonian systems have different expressions.

### 3. Examples.

In this section, and in the application of Theorems 2.1 and 2.4 as well as of Corollaries 2.2 and 2.3, we analyze the nonintegrability of the Hamiltonian systems corresponding to the following differential equations:

1. Hill–Schrödinger equation: $\ddot{x} = k(t)x$.
2. Sitnikov problem: $\ddot{x} = \frac{(1 - \epsilon \cos t)x}{(x^2 + r^2(t)^{3/2})}$, $r(t) = \frac{1 - \epsilon \cos t}{2}$.
3. Painlevé II equation: $\ddot{x} = 2x^3 + tx + \alpha$.
4. An algebraic toy model: $\ddot{x} = -\frac{1}{x^2} - \frac{t}{x} + \alpha$.

In order to analyze normal variational equations, a standard procedure is using Maple, and especially the commands `dsolve` and `kovacicsols`. Whenever the command `kovacicsols` yields an output “[ ]”, it means that the second-order linear differential equation being considered has no Liouvillian solutions, and thus its Galois group is virtually nonsolvable. For equations of the form $\ddot{y} = ry$ with $r \in \mathbb{C}(x)$, the only virtually nonsolvable group is $\text{SL}_2(\mathbb{C})$. In some cases, moreover, `dsolve` makes it possible to obtain the solutions in terms of special functions such as Airy functions, Bessel functions, and hypergeometric functions, among others [2]. There is a number of second-order linear equations whose coefficients are not rational and whose solutions Maple cannot find by means of the commands `dsolve` and `kovacicsols` alone; this problem, in some cases, can be solved by the stated algebrization procedure.

#### 3.1. Hill–Schrödinger equation $\ddot{x} = k(t)x$.

This example corresponds to Corollary 2.2 for $k > 0$, $\epsilon \gg 0$, $P_n$ a polynomial of degree $n$ with $P_n(0) = 1$, and $k(t)$ given by

\[
\begin{align*}
k(t) &= ke^{-t}, & k(t) &= kP_n(et), \\
k(t) &= k(1 + \sinh(et)), & k(t) &= k(1 + \sin(et)), \\
k(t) &= k(1 + \cosh(et)), & k(t) &= k(1 + \cos(et)).
\end{align*}
\]
$X_{\tilde{H}}$ is nonintegrable by means of rational first integrals.

The integrability of equation $\ddot{x} = k(t)x$ for these examples has been deeply analyzed in [3].

### 3.2. The Sitnikov problem.

The Sitnikov problem is a symmetrically configured restricted three-body problem in which two primaries with equal masses move in ellipses of eccentricity $e$ in a plane $\pi_1$, and an infinitesimal point mass moves along the line $\pi_1^\perp$. See [6], [12], [25], [27], [37], [38], [16] for more details. The motion of the infinitesimal point mass is given by the differential equation

$$
\ddot{x} + \frac{z}{(r^2(t) + z^2)^{3/2}} = 0,
$$

where $z = z(t)$ is the distance from the infinitesimal mass point to the plane of the primaries and $r(t)$ is half the distance of the primaries,

$$
r(t) = \frac{1 - e \cos E(t)}{2},
$$

where the eccentric anomaly $E(t)$ is the solution of the Kepler equation

$$
E = t + e \sin E
$$

and $e$ is the eccentricity of the ellipses described by the primaries. We will assume $0 \leq e \leq 1$ through subsections 3.2 and 3.3. The Hamiltonian linked to the system is

$$
H = \frac{v^2}{2} - \frac{1}{(z^2(t) + r^2(t))^{1/2}},
$$

provided $v$ stands for $\dot{z}$ in the corresponding equations.

Since (3.3) cannot be solved in explicit form, attempts at a Hamiltonian formulation of (3.1), whether exact or approximate, require one of at least two options: looking for an exact Hamiltonian formulation by means of a change of variables, which we will do in the next paragraph, and searching an approximate Hamiltonian formulation, which will be done in subsection 3.3.

Let us now find a Hamiltonian linked to (3.1). We may express $r(t)$ as

$$
r(t) = (R \circ \varphi)(t) := \frac{1 - e^2}{2(1 + e \cos \varphi(t))},
$$

where $\varphi$, the true anomaly, is a solution of

$$
\frac{d \varphi}{dt} = \frac{(1 + e \cos \varphi)^2}{(1 - e^2)^{3/2}} = \frac{\sqrt{1 - e^2}}{4R^2(\varphi)},
$$

and we may follow the procedure introduced in [38] (see also [16]) by taking $\varphi$ as the new independent variable and $x = \frac{\dot{z}}{2r(\varphi)}$ as the new dependent variable. Writing $t$ once again to denote $\varphi$, we have the following differential equation:

$$
\dot{x} = f(x,t) := -\frac{e \cos t + \left(\frac{1}{4} + x^2\right)^{-3/2}}{1 + e \cos t}x,
$$

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clearly amenable to the hypotheses in Theorem 2.1 and in the first paragraph of section 2. Defining \( q_1 = x, q_2 = t, \) and \( p_1 = \dot{x}, \) the autonomous Hamiltonian system corresponding to (3.4) is given by

\[
\hat{H}_e = H_e + p_2 := \frac{p_1^2}{2} + \frac{eq_t^2 \cos q_2 - 4 \left(1 + 4q_t^2\right)^{-1/2}}{2 \left(1 + e \cos q_2\right)} + p_2,
\]

always assuming \( e \in [0, 1]. \)

The circular Sitnikov problem \( \hat{H}_0 \) is meromorphically integrable in the sense of Liouville–Arnold and can be solved using elliptic integrals. The nonintegrability for \( e = 1 \) was first studied by means of straight Morales–Ramis theory in [26, sect. 5] (see also [22, sect. 5.3]); we will now extend the proof of meromorphic nonintegrability therein to one for every \( 0 < e \leq 1 \) by using Theorem 2.1.

The NVE \( \Gamma \) derived from the Hamiltonian system given by (3.5) is

\[
\dot{\xi} = \left(\frac{e(4q_1^2 + 1)^{5/2} \cos q_2 - 64q_1^2 + 8}{(e \cos q_2 + 1)(4q_1^2 + 1)^{5/2}}\right) \xi.
\]

Taking \( q_1 \equiv 0 \) we have a solution \( \Gamma = \{z(t) = (0, t, 0, \frac{2}{1 + e \cos t})\} \) of \( X_{\hat{H}_e} \) along which the NVE \( \Gamma \) is given by

\[
\dot{\xi} = \left(\frac{e \cos t + 8}{e \cos t + 1}\right) \xi,
\]

which is algebrizable, through the change of variable \( \tau = \cos t, \) into

\[
\frac{d^2 \xi}{d\tau^2} - \left(\frac{\tau}{1 - \tau^2}\right) \frac{d\xi}{d\tau} - \left(\frac{e \tau + 8}{(e \tau + 1)(1 - \tau^2)}\right) \xi = 0.
\]

This equation can be transformed into the differential equation

\[
\frac{d^2 \zeta}{d\tau^2} = \left(\frac{5e \tau^3 + 33\tau^2 - 2e \tau - 30}{(e \tau + 1)4(1 - \tau^2)^2}\right) \zeta
\]

by means of \( \xi = \frac{\zeta}{\sqrt{1 - \tau^2}}. \) Equations (3.6) and (3.7) are integrable in terms of Liouvillian solutions only if \( e = 0, \) whereas for \( e \neq 0 \) their solutions are given in terms of nonintegrable Heun functions if \( e \neq 1 \) and nonintegrable hypergeometric functions if \( e = 1, \) i.e., their Galois groups are virtually nonsolvable and hence virtually nonabelian; furthermore, they have a regular singularity at infinity, which by Theorem 1.2 implies the nonintegrability of \( X_{\hat{H}} \) for \( e \in (0, 1] \) by means of meromorphic first integrals. In particular, the Galois group of (3.7) is exactly \( SL_2(\mathbb{C}) \).

3.2.1. Numerical results for the Sitnikov problem. The author is indebted to Sergi Simon in what concerns the following subsection, including the figures. Acknowledgments are also due to Carles Simó for further specific suggestions.

Let \( \Sigma = \{\sin (q_2) = 0\}. \) The six figures in this paper show Poincaré sections of the flow with respect to \( \Sigma, \) projected on the \((q_1, p_1)\) plane, for the Hamiltonian system \( X_{\hat{H}_e} \) obtained.
Figure 1. Poincaré section \( \sin q_2 = 0 \) for the Sitnikov problem: \( e = 0 \) (figure: Sergi Simon).

from (3.5). As may be easily deduced from said Hamiltonian, all sections are symmetrical with respect to the \( q_1 \) and \( p_1 \) axes. Different numbers of initial conditions are used for the sake of clarity.

Figure 1 corresponds to \( e = 0 \). In keeping with what was said after (3.5), the whole subset of \( \Sigma \) transversal to the flow sheds concentric tori (ostensibly, the intersections of the invariant Liouville–Arnold tori with \( \Sigma \)), a typical sign of integrability; the tori shown are only a selection of those therein, as the actual area foliated by them is larger.

A number of these invariant tori break down upon the slightest increase in \( e \), and in the ensuing figures the two most interesting features are those invariant sets (usually called \textit{KAM tori}) whose intersection with \( \Sigma \) prevails in the form of Jordan curves, and the zones of chaotic behavior between them. Sparse zones of the section will account for chaotic zones as well for \( e > 0 \). Figures 2 and 3 show two different close-up views for the Poincaré section corresponding to \( e = 0.01 \). The latter figure is actually a detail of the “island” of tori appearing at the right of the general section. For \( e = 0.1 \), Figures 4 and 5 are, respectively, a general view of the section and a close-up of one of the islands appearing at each side of the central area. As for \( e = 0.4 \), Figure 6 is an enlarged view of one of the two islands appearing at each side of a central area.

3.3. The approximate Sitnikov problem. As said in subsection 3.2, we now consider an approximation of the Sitnikov problem; see [15] and [12] for more details. As opposed to \textit{meromorphic} nonintegrability, we will prove nonintegrability by means of \textit{rational} first integrals.
Figure 2. Poincaré section $\sin q_2 = 0$ for the Sitnikov problem, $e = 0.01$ (figure: Sergi Simon).

The fact that $\varphi (t) = t + O (e)$ yields $r (t) = \frac{1 - e \cos t}{2} + O (e^2)$, and thus the Hamiltonian in (3.3) becomes

$$H = \frac{v^2}{2} - \frac{1}{\sqrt{z^2 + \frac{1}{4}}} - e \cos t \left( \frac{z^2 + \frac{1}{4}}{z^2 + \frac{1}{4}} \right)^{3/2} + O (e^2)$$

whenever $e \approx 0$. In particular, the first-order approximation of this asymptotic expansion in $e$ yields the Hamiltonian

(3.8)

$$H = \frac{v^2}{2} - \frac{1}{\sqrt{z^2 + \frac{1}{4}}} - e \cos t \left( \frac{z^2 + \frac{1}{4}}{z^2 + \frac{1}{4}} \right)^{3/2}.$$

Considering $q_1 = x$, $q_2 = t$, and $p_1 = v$, the autonomous Hamiltonian system corresponding to this equation is given by

(3.9)

$$\hat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - e \frac{2 \cos q_2}{(4q_1^2 + 1)^{3/2}} - \frac{2}{\sqrt{4q_1^2 + 1}}.$$

corresponding to the Hamiltonian in Theorem 2.1:

$$f(x, t) = - \frac{8x}{(4x^2 + 1)^{3/2}} - e \frac{24x \cos t}{(4x^2 + 1)^{3/2}}.$$
The NVE for the Hamiltonian (3.9) is given by

$$\ddot{\xi} = \left( e^{24(16q_1^2 - 1)\cos q_2 \frac{1}{(4q_1^2 + 1)^2}} + \frac{8(8q_1^2 - 1)}{(4q_1^2 + 1)^2} \right) \xi.$$  

Its general solution may be expressed as

$$\xi(t) = K_1 C \left( 32, 48e, \frac{t}{2} \right) + K_2 S \left( 32, 48e, \frac{t}{2} \right),$$

where the Mathieu even (resp., odd) function $C(a, q, t)$ (resp., $S(a, q, t)$) is defined as the even (resp., odd) solution to $\ddot{y} + (a - 2q \cos (2t))y = 0$ [2, Ch. 20].

Taking $q_1(t) = 0$, we can see that $z(t) = (0, t, 0, 2e \cos t + 2)$; hence, defining $z(t) = (0, t, 0, 2e \cos t + 2)$ and $\Gamma = \{z(t)\}$, the operator linked to NVE$_\Gamma$ is given by

$$\ddot{\xi} = (-24e \cos t - 8) \xi,$$

which is algebrizable (see Theorem 1.1) through the change of variables $\tau = \cos t$ into

$$(3.10) \quad \frac{d^2 \xi}{d\tau^2} - \left( \frac{\tau}{1 - \tau^2} \right) \frac{d\xi}{d\tau} + \left( \frac{24e\tau + 8}{1 - \tau^2} \right) \xi = 0.$$  

Now, this equation can be transformed into the differential equation

$$(3.11) \quad \frac{d^2 \zeta}{d\tau^2} = \left( \frac{96e\tau^3 + 31\tau^2 - 96e\tau - 34}{4(1 - \tau^2)^2} \right) \zeta, \quad \xi = \frac{\zeta}{\sqrt{1 - \tau^2}}.$$
Equations (3.10) and (3.11) are integrable in terms of Liouvillian solutions only if $e = 0$, since for $e \neq 0$ their solutions are given in terms of nonintegrable Mathieu functions; hence their Galois groups are virtually nonsolvable and thus virtually nonabelian; furthermore, they have an irregular singularity at infinity, implying rational nonintegrability for the Hamiltonian field $X_{\hat{H}}$ with $\alpha = 1$ by virtue of Theorem 1.2. In particular, the Galois group of (3.11) is exactly $\text{SL}_2(\mathbb{C})$.

Equations (3.10) and (3.11) have been deeply analyzed in [3] using the Hamiltonian change of variables $\tau = e^{it}$, obtaining the same result presented here.

3.4. Painlevé II equation: $\ddot{x} = 2x^3 + tx + \alpha$. The nonintegrability of the second Painlevé equation for integer $\alpha$ was proved by Morales-Ruiz in [23] and later by Stoyanova and Christov in [29]. They used only the Hamiltonian (3.14).

Defining $q_1 = x$, $q_2 = t$, $p_1 = y$, and $\alpha \in \mathbb{C}$, the autonomous Hamiltonian system corresponding to this equation can be given by any of the following three functions:

\begin{align*}
\hat{H} &= H + p_2, \quad H = \frac{p_1^2}{2} - \frac{q_1^4}{2} - q_2^2 + \alpha q_1, \\
\hat{H} &= H + p_2, \quad H = \frac{p_1^2}{2} - \frac{1}{2} \left( q_1^2 + q_2^2 \right)^2 - \alpha q_1, \\
\hat{H} &= H + p_2, \quad H = \frac{p_1^2}{2} - \left( q_1^2 + q_2^2 \right) p_1 - \left( \alpha + \frac{1}{2} \right) q_1,
\end{align*}

where (3.12), (3.13), and (3.14) correspond to the Hamiltonian of Theorem 2.1 ($f(x,t) =$...
2x^3 + tx + \alpha), Corollary 2.3 \((g(x) = x^2\) and \(a(t) = t/2)\), and Theorem 2.4 \((g(x) = x^2\) and \(a(t) = t/2\), respectively. The Hamiltonian system for Painlevé II, studied in [23, 29], corresponds precisely to Hamiltonian (3.14). The NVE\(\Gamma\) for these Hamiltonians is given by

\[ \ddot{\xi} = (6q_1^2 + q_2) \xi. \]

Taking \(\alpha = 0\) and \(q_1(t) = 0\), we have particular solutions \(z(t) = (0, t, 0, 0)\), \(z(t) = (0, t, 0, t^2/8)\), and \(z(t) = (0, t, t/2, t^2/8)\), respectively, for the Hamiltonians (3.12), (3.13), and (3.14); hence NVE\(\Gamma\) is given by \(\dot{\xi} = t\xi\), the so-called Airy equation [2, sect. 10.4.1], which has an irregular singularity at infinity and is not integrable through Liouvillian solutions; i.e., its Galois group is SL\(_2(\mathbb{C})\) and not virtually abelian; thus, by Theorem 1.2, the Hamiltonian field \(X_{\hat{H}}\) with \(\alpha = 0\) is not integrable through rational first integrals.

Now, for \(\alpha = 1\) and \(q_1(t) = -1/t\), the integral curve \(z(t)\) is given by

\[
\begin{align*}
\left(\frac{1}{t}, t, \frac{1}{t^2}, -\frac{1}{2t}\right), & \quad \left(\frac{1}{t}, t, \frac{1}{t^2}, -\frac{1}{2t} + \frac{t^2}{8}\right), & \quad \text{and} & \quad \left(-\frac{1}{t}, t, \frac{2}{t^2} + \frac{t}{2} - \frac{1}{t} + \frac{t^2}{8}\right),
\end{align*}
\]

respectively, for the Hamiltonians (3.12), (3.13), and (3.14), so that NVE\(\Gamma\) is given by

\[ \ddot{\xi} = \left(\frac{6}{t^2} + t\right) \xi, \quad \Gamma = \{z(t)\}, \]

Figure 5. Poincaré section for \(e = 0.1\) (figure: Sergi Simon).
whose general solution is

$$\xi(t) = \sqrt{t} \left[ K_1 I_{-5/3} \left( \frac{2t^{3/2}}{3} \right) + K_2 I_{5/3} \left( \frac{2t^{3/2}}{3} \right) \right],$$

$I_{\alpha} = 2^{-\alpha} t^{\alpha} \left( \frac{1}{\Gamma(1+\alpha)} + \frac{t^2}{2! \Gamma(2+\alpha)} + O(t^4) \right)$ being, for each $\alpha$, the modified Bessel function of the first kind, i.e., the solution to $t^2 \ddot{y} + t \dot{y} - (t^2 + \alpha^2) y = 0$ [2, sect. 9.6].

The normal variational equation has an irregular singularity at infinity and is not integrable through Liouvillian functions because its solutions are given in terms of nonintegrable Bessel functions (see [23, 29]); i.e., its Galois group is $\text{SL}_2(\mathbb{C})$, which is not virtually abelian; again by the Morales–Ramis theorem, Theorem 1.2, the Hamiltonian field $X_{\hat{H}}$ with $\alpha = 1$ is not integrable through rational first integrals.

### 3.5. An algebraic toy model: $\ddot{x} = -\frac{1}{4x^2} - \frac{1}{x^2} + \alpha$.

Considering $q_1 = x$, $q_2 = t$, $p_1 = y$, and $\alpha \in \mathbb{C}$; the autonomous Hamiltonian systems corresponding to this equation are given by

$$(3.15) \quad \hat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - \frac{1}{8q_1^2} \frac{q_2}{q_1} - \alpha q_1,$$

$$(3.16) \quad \hat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - \frac{1}{2} \left( \frac{1}{2q_1} + 2q_2 \right)^2 - \alpha q_1,$$

$$(3.17) \quad \hat{H} = H + p_2, \quad H = \frac{p_1^2}{2} + \left( \frac{1}{2q_1} + 2q_2 \right) p_1 - (\alpha + 2) q_1.$$
Equations (3.15), (3.16), and (3.17) correspond to Theorem 2.1 \( f(x,t) = -\frac{1}{4t}, -\frac{1}{2^x} + \alpha \), Corollary 2.3 \( g(x) = -\frac{1}{2^x} \) and \( a(t) = -2t \), and Theorem 2.4 \( g(x) = -\frac{1}{2^x} \) and \( a(t) = -2t \), respectively. The NVE\( \Gamma \) for all three is given by

\[
\ddot{\xi} = \left( \frac{3}{4q_1^2} + \frac{2q_2}{q_1^3} \right) \xi.
\]

Now, for \( \alpha = 1 \) and \( q_1(t) = \sqrt{t} \), the integral curve \( z(t) \) is given by

\[
\left( \sqrt{t}, t, \frac{1}{2\sqrt{t}}, 2\sqrt{t} \right), \quad \left( \sqrt{t}, t, \frac{1}{2\sqrt{t}}, 2t^2 + 2\sqrt{t} \right), \quad \text{and} \quad \left( \sqrt{t}, -2t, 2t^2 \right),
\]

respectively, for the Hamiltonians (3.15), (3.16), and (3.17), rendering NVE\( \Gamma \) equal to

(3.18)

\[
\ddot{\xi} = \left( \frac{3}{4t^2} + \frac{2}{\sqrt{t}} \right) \xi,
\]

having a solution

\[
\xi_1 = -\frac{3t^{3/2}}{2} \, _0F_1 \left( \frac{7}{3}; \frac{8t^{3/2}}{9} \right) = -\frac{3}{2}t^{3/2} - \frac{4}{7}t - \frac{8}{105}t^{9/2} + O(t^6),
\]

\( _0F_1 (\cdot ; a; t) = \lim_{q \to -\infty} _1F_1 \left( q; \frac{a}{q} \right) = \sum_{n=0}^{\infty} \frac{a^n}{(a)_n n} \) being the confluent hypergeometric limit function [2, Ch. 13], and an independent new solution \( \xi_2 = \xi_1 \frac{\Delta \xi}{\xi_1^2} \), satisfying

\[
\xi_2 = \frac{1}{3\sqrt{t}} - \frac{8t}{9} - \frac{16}{27}t^{5/2} + O(t^4).
\]

As is the case for the rest of the normal variational operators appearing in this paper, our knowledge of the exponents around 0 of a fundamental set of solutions (in this case, \( \xi_1 \) and \( \xi_2 \)), coupled with the basic result on factorization obtained in [7, Th. 8 (Ch. 5)] (see also [8, Criterion 1]), would suffice to prove nonintegrability. Here, however, we will restrict our attention to Theorems 2.1 and 2.4 and Corollary 2.3.

Equation (3.18) is algebrizable (Theorem 1.1), through the change of variables \( \tau = \sqrt{t} \) into

(3.19)

\[
\frac{d^2\xi}{d\tau^2} - \left( \frac{1}{\tau} \right) \frac{d\xi}{d\tau} - \left( \frac{8\tau^3 + 3}{\tau^2} \right) \xi = 0;
\]

now, this equation can be transformed into the differential equation

(3.20)

\[
\frac{d^2\zeta}{d\tau^2} = \left( \frac{32\tau^3 + 15}{4\tau^2} \right) \zeta, \quad \xi = \zeta \sqrt{\tau}.
\]

Equations (3.19) and (3.20) have an irregular singularity at \( t = \infty \) and are not integrable through Liouvillian solutions due to the presence of Bessel functions; i.e., their Galois groups are virtually nonsolvable and therefore virtually nonabelian, Theorem 1.2 once again settling rational nonintegrability for \( \alpha = 1 \). In particular, the Galois group of (3.20) is exactly \( \text{SL}_2(\mathbb{C}) \).
4. Final remarks: Open questions and future work. This paper is the starting point of a project in which the author is involved. The following questions arose during our work:

- In [23, 29] it was proven that the autonomous Hamiltonian system related to Painlevé II is nonintegrable for every $\alpha \in \mathbb{Z}$. Is this also true for (2.6)?
- Does the integrability of (2.6) for arbitrary $\alpha \in \mathbb{Z}$ depend on the choice of $g(x)$ and $a(t)$?
- Assuming the above question has an affirmative anser, in what manner can the choice and form of $g(x)$ and $a(t)$ ensure nonintegrability for every $\alpha \in \mathbb{Z}$? And for every $\alpha \in \mathbb{C}$?
- Is it possible to find transversal sections of the flow, and thus Poincaré maps, for either $\dot{H}$ or the algebrized equation, even in the absence of nontrivial numerical monodromies? Do Stokes multipliers contribute to the answer in a significant manner?

Among our next goals, the analysis of the following items is due further immediate research:

- the application of Morales–Ramis theory to higher variational equations of nonautonomous Hamiltonian systems;
- differential equations in the form $\ddot{x} = f(x, \dot{x}, t)$;
- the rest of the Painlevé equations: Casale in [10] analyzed Painlevé I, and Horozov and Stoyanova in [17] analyzed some particular cases for Painlevé VI;
- the theoretical aspects of nonautonomous Hamiltonian systems such as their geometry and the feasibility of an analogue to Liouville–Arnold theory;
- the nonintegrability of nonautonomous Hamiltonian systems with two and a half degrees of freedom;
- specific examples of nonautonomous Hamiltonian systems related to control theory, as well as others related to celestial mechanics, such as the restricted three- and four-body problems and Hénon–Heiles systems [4], [13].
- the exact relation, perhaps causal, between separatrix splitting [24] and nonintegrability, whether rational or meromorphic.

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